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Constriction resistance and the real area of contact

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Abstract. The relation between the area of contact and the constriction resistance which holds for a single circular contact spot is widely used in electric contact theory, although the normal mode of contact is by a large number of microcontacts. A method of finding the resistance of a cluster of microcontacts is derived, and it is shown that the resistance may be regarded as the sum of the parallel resistance of the microcontacts and an interaction term often related to the extent of the cluster and not to the number or size of the individual contacts. The resistance is often close to that found by assuming that the entire area covered by the cluster is a single conducting spot.

The known agreement between areas of contact found from resistance measurements and by other methods is therefore puzzling—until it is realized that the other methods also give only an apparent area: the real area of contact in, for example, a Brinell indentation is a small fraction of the area of the indentation. Thus from the point of view of electric contact theory the system is self-consistent, although the real area of contact is now seen to play no part in it: the implications for the theory of friction are more profound.

1. Introduction

When two large conductors touch over a small circular area, there is a constriction resistance between them of $\rho/2a$, where a is the radius of the contact circle. This equation is very widely used in the design and study of electrical contacts. There is evidence, however, that contact between metals rarely, if ever, occurs in this way. For example, figure 1 shows a surface profile across a Brinell hardness indentation. The apparent contact area, used to assess the hardness, is subdivided into a multitude of microcontacts.

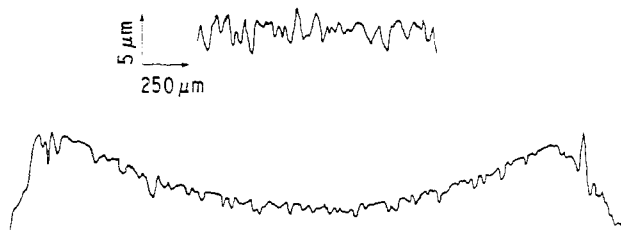


Figure 1. Talysurf profile of Brinell impression in annealed bead-blasted gold block. Most of the curvature of the indentation has been cut out using the Talysurf radius attachment. The upper profile shows part of the undeformed surface.

Contact between nominally flat surfaces is known to occur as a number of clusters of microcontacts (Dyson and Hirst 1954), the position of the clusters being determined by the large-scale waviness of the surface, and the microcontacts by the small-scale surface roughness.

It is clear that the constriction resistance will be partly determined by the number and size of the microcontacts and partly by their grouping into clusters. The simplest case is that of a large number of small equal spots distributed uniformly and densely over a

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circular area. The resistance is then

$$R = \rho \left(\frac{1}{2na} + \frac{1}{2\alpha} \right) \quad (1)$$

where a is the radius of a spot, and α the radius of the cluster (Holm 1929). However, this equation is rarely used—a notable exception being Cocks (1954 Ph.D. Thesis, University of Reading)—perhaps because it does not occur in Holm's more accessible publications (e.g. Holm 1958). The present paper deals more generally with the question of the resistance of a cluster of contact spots, and suggests a wider interpretation of Holm's result.

2. Mathematical approach

Current-flow problems are completely analogous to electrostatic charge distribution problems: the current flow through a given contact is replaced by the charge on a conductor shaped like the contact spot. The method which follows uses the fact that we can (approximately) determine the potential due to given charges on a set of conductors; we then apply the condition that these potentials must be equal and obtain a set of simultaneous equations determining the charges. The procedure in electrostatics is well-known (see for example Smythe 1950); it is convenient in what follows to paraphrase it in current-flow terminology.

We consider a single cluster which is small compared to the size of the nominal contact area and to the distance from any other cluster. The cluster consists of n microcontacts. The current flowing through the k th contact is I_k . We require first the potentials at the contacts in terms of the currents I_k . Let b_{ij} be the potential difference between contact i and infinity due to unit current through contact j , the other contacts being present (i.e. being equipotentials) but passing no current. We will refer to b_{ij} ($i \neq j$) as the mutual resistance of contacts i and j , and to b_{ii} as the self-resistance of contact i . The self-resistance is not exactly the same as the resistance of the contact if there were no other contacts since the presence of additional equipotential areas causes a redistribution of current and so a change in resistance; but in this section such effects will be neglected: a later section will consider the errors so introduced.

We consider the situation in one of the contacting bodies only. Thus, for a circular contact of radius a_i , the self-resistance will be $\rho/4a_i$ (e.g. Smythe 1950).

To a first approximation the mutual resistance of two contacts can be found by assuming them to be point contacts: in fact, the potential at a distance s from contact j is $\rho I_j/2\pi s$ (corresponding to $\frac{1}{2}Q/s$ in the electrostatic case). Thus $b_{ij} \sim \rho/2\pi s_{ij}$ where s_{ij} is the distance between the contacts. As before, the effect of the higher order corrections will be postponed to a later section (§2.3).

The total potential at the i th contact is the sum of the potentials due to the separate currents, i.e.

$$\phi_i = \sum_j b_{ij} I_j. \quad (2)$$

For an array of circular spots we have approximately

$$\phi_i = \frac{\rho}{4} \left(\frac{I_i}{a_i} + \frac{2}{\pi} \sum_{j \neq i} \frac{I_j}{s_{ij}} \right).$$

For a cluster at the interface between two large conductors of the same material the potentials must all be equal, with a value of half the potential difference applied across the two conductors[†], i.e.

$$\frac{U}{2} = \frac{\rho}{4} \left(\frac{I_i}{a_i} + \frac{2}{\pi} \sum_{j \neq i} \frac{I_j}{s_{ij}} \right). \quad (3)$$

[†] With some obvious minor changes, the theory also applies to the high resistance member of contact between conductors of very different resistivities.

In principle we can solve this set of equations to find the currents I_i and hence the total current and the resistance. This has been done for some particular clusters with a limited number of contacts and the results are given below: we shall first, however, obtain an approximate formula which avoids this process and so is more generally useful.

2.1. Approximate formulae for the resistance

It is well known that the actual distribution of current in a conductor is the one which makes the rate of heat production a minimum. The significance of this result here is that a minimum is a stationary value, and so is insensitive to small changes in the currents. Thus a reasonable estimate of the currents will give a better estimate of the resistance.

The rate of heat production is

$$W = 2 \sum_i \phi_i I_i = 2 \sum_i \sum_j b_{ij} I_i I_j$$

on substituting for ϕ_i from equation (2): the factor 2 comes from the two symmetrical conductors.

When the microcontacts are far apart they will act independently and the current through each will be inversely proportional to its self-resistance b_{ii} , i.e. $I_i \sim \lambda/b_{ii}$. Assuming this to be approximately true in the general case, we have

$$I = \sum I_i = \lambda \sum \frac{1}{b_{ii}}, \quad W = 2\lambda^2 \sum_i \sum_j \frac{b_{ij}}{b_{ii}b_{jj}}$$

and so

$$R = \frac{W}{I^2} = 2 \left(\sum_i \sum_j \frac{b_{ij}}{b_{ii}b_{jj}} \right) \left/ \left(\sum \frac{1}{b_{ii}} \right)^2 \right.$$

Substituting the values for an array of circular contacts gives

$$b_{ii} = \rho/4a_i, \quad b_{ij} = \rho/2\pi s_{ij} \quad (i \neq j)$$

and

$$R = \frac{\rho}{2\sum a_i} + \frac{\rho}{\pi} \left(\sum_{i \neq j} \frac{a_i a_j}{s_{ij}} \right) \left/ (\sum a_i)^2 \right. \quad (4)$$

This expression is quite simple to use since the terms are known and the evaluation involves only summations; the solution of the simultaneous equations, which rapidly becomes impracticable as the number of spots increases, has been circumvented. From the minimum property of the heat production it follows that this will give an overestimate of the resistance.

A further approximation is very useful. If we assume there is no correlation between the size of a contact and its position, we can approximately replace the individual radii in the double summation by their mean, and get

$$R \sim \frac{\rho}{2\sum a_i} + \frac{\rho}{\pi n^2} \sum_{i \neq j} \sum \frac{1}{s_{ij}} \quad (5)$$

This, of course, is exact if the contacts are all the same size: if the sizes vary (but are uncorrelated with position) the error is small and decreases as the number of contacts in the cluster increases. The error, unlike that of equation (4), can be of either sign: however, the value is usually again an overestimate. The first term in both approximations is the usual expression for the resistance of a number of contact spots in parallel; it ignores their positions. To this is added a term which expresses the interaction between the spots; in formula (4) this term involves the position and the sizes of the spots; in formula (5) it depends only on their positions. The magnitude of this term can be estimated quite easily: for a cluster of linear dimension l the mean distance between a pair of microcontacts will be of the order $\frac{1}{2}l$, and the double summation will be approximately $n(n-1)2/l$. Hence for n large, the second term becomes $2\rho/\pi l$, and so is comparable with the first term whenever

$$l \sim \sum a_i = n\bar{a}$$

i.e. when the linear dimensions of the cluster are comparable with the sum of the linear dimensions of the contacts.

2.2. Application to a typical cluster

To determine the accuracy of the practical resistance formulae (4) and (5), a number of arrays were constructed, their resistances found by the solution of the simultaneous equations (3), and compared with the approximate answers. The resistances given are purely numerical: they must be multiplied by the resistivity expressed in appropriate units; for example, if the array dimensions are regarded as being in micrometres, the resistivity must be in ohm micrometres to give the resistance in ohms.

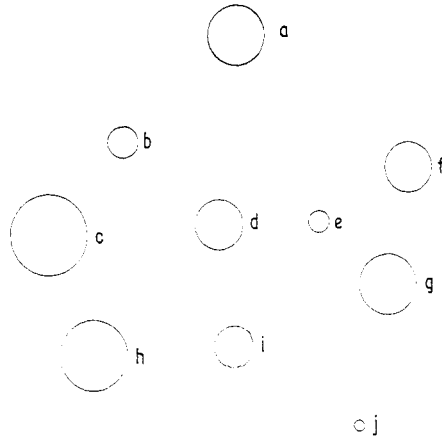


Figure 2. Typical random array. The individual currents are tabulated below.

Contact	a	b	c	d	e	f	g	h	i	j
Radius a	0.6	0.3	0.8	0.5	0.2	0.5	0.6	0.7	0.4	0.1
Current I	0.836	0.317	1.036	0.513	0.178	0.638	0.758	0.888	0.445	0.129
Ratio I/a	1.39	1.06	1.29	1.03	0.89	1.28	1.26	1.27	1.11	1.29

Figure 2 shows a typical random array. The full solution gave a resistance of 0.174; the two approximate formulae (4) and (5) gave respectively 0.175 and 0.179, i.e. errors of 0.7% and 2.9%, which are quite acceptable. The resistance for the same contacts acting independently was only 0.106, showing the importance of the interaction term. It is interesting to examine the way the current divides between the contacts, which is known from the full solution (see caption, figure 2). The contacts are not acting independently since the currents are not proportional to the radii. Contact a carries an unduly high current while e carries an unduly low one; both of which are easily understood from the figure: a is the most isolated contact and e the most enclosed.

If we keep the spot sizes unchanged but double the distances between them, the contacts should be more independent. This is confirmed by the results: the resistance falls to 0.141 and the errors in the approximate formulae become 0.2% and 1.6% respectively. The converse occurs if we halve the distances between the contacts, again keeping the sizes unchanged. The resistance rises to 0.240 and the errors in the approximate formulae become 2.1% and 5.3%. A further decrease in separation would be meaningless, since the contacts have already begun to overlap.

Results on some 70 arrays show that half the errors are less than 1% using either approximate formula: larger errors, up to 5%, occur only in extreme cases used to test the method, such as the array containing touching spots just described. Only one error appreciably over 5% has ever been found: an error of 9.4% occurred with formula (5) for a set of spots placed as in figure 2, but with nine small spots (b to j, all of radius 0.5) and one large one (a, radius 2.0). In any real case the error will be small.

2.3. The accuracy of the basic method

The basic equations (3) from which the individual currents and hence the overall resistance are found are only exact in the limits as the microcontacts become smaller or infinitely separated. For finite microcontacts there are errors which depend on the separa-

tion and on the shape. It is shown in the Appendix that for a pair of contacts the errors in the self and mutual resistances are of order a^3/s^4 and a^2/s^3 respectively. In electrostatic terminology the former represents a dipole effect, since the charges induced on a second contact by the charge on the first contact affect the potential of the first; and the latter a quadrupole effect, since the charge is spread over an area instead of being localized at a point. For a cluster of n contacts we get a symmetric quadrupole interaction between every pair of contacts, and unsymmetric dipole interactions involving every triplet: the charge on a induces a dipole on b which affects the potential of c , and so on. Clearly it is not in general practicable to take all these into account. One case where it can be done is for a cluster consisting of six contacts symmetrically round a seventh (see figure 4(a), §3.2.) The results in table 1 give the individual currents and the overall resistances (taking the radii to be 1.0) using (i) terms up to s^{-1} , (ii) terms up to s^{-3} (quadrupole effects), (iii) terms up to s^{-4} (dipole effects).

Table 1. Analysis of hexagonal cluster (figure 4(a))

Approximation	Current in centre contact	Current in outer contact	Overall resistance
First order	0.3259	0.8765	0.1790
Quadrupole	0.2122	0.8641	0.1853
Dipole	0.4254	0.8587	0.1743

The results show two important features. The dipole effects more than outweigh the quadrupole effects, so that there is no justification for including one but not the other or, indeed, for excluding even higher terms. Secondly, and more importantly, the overall resistance is insensitive to the inclusion of the higher terms. It varies only by 3% in an example which has the maximum possible number of touching neighbours. Remembering that the interactions decrease as a^2/s^3 and a^3/s^4 respectively, it is clear that with even a small separation between the contacts the higher order effects will be negligible. Thus, for example, for a hexagonal cluster with a centre-to-centre distance of two diameters, the three resistance values are 0.2529, 0.2538 and 0.2521 which vary by only 0.3%.

3. Physical interpretation

3.1. Holm's equation

It is possible to give a geometrical interpretation to the interaction term. If we introduce the mean contact radius \bar{a} , equation (5) becomes

$$R = \frac{\rho}{2n\bar{a}} + \frac{\rho}{\pi n^2} \sum_{i \neq j} \sum_{ij} \frac{1}{s_{ij}}.$$

This may be used to obtain the resistance of any array of contact spots, even if their positions are only known statistically. Consider for example a cluster comprising a very large number of spots uniformly distributed with q per unit area within a circle of radius a . The evaluation of the sum of all the reciprocal distances involves some rather tedious integrations (see Timoshenko and Goodier 1951): the result is $16n^2/3\pi a$. Thus,

$$R = \rho \left(\frac{1}{2n\bar{a}} + \frac{16}{3\pi^2 a} \right). \quad (6)$$

Since $16/3\pi^2 = 0.5404$, this is seen to be close to Holm's result

$$R = \rho \left(\frac{1}{2na} + \frac{1}{2a} \right). \quad (1)$$

The difference is informative, for the term $(16/3\pi^2) \rho/a$ is the resistance of a circular contact area of radius a when the current is constrained to flow uniformly through it (Rayleigh 1871). When the current is free to distribute itself as it likes the resistance falls to the

usual $\rho/2a$. It is clear physically that both answers are correct, equation (1) being appropriate when the self-resistance is small, and (6) when the self-resistance is dominant. This can be tested mathematically by taking an array of contacts covering an approximately circular area (figure 3) and varying the size of the contacts while keeping their positions unchanged. Table 2 shows how the interaction, defined as $R - (1/2na)$, varies as the self-resistance $1/2na$ is decreased (R being calculated by the simultaneous equation method).

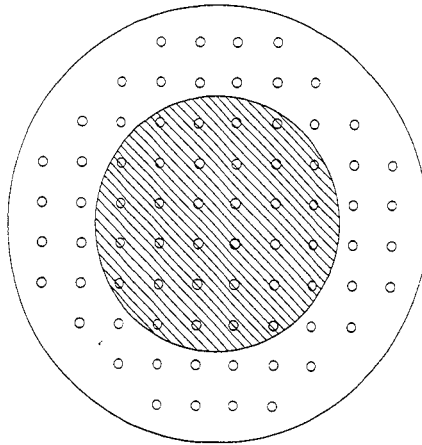


Figure 3. Resistance of a regular array of contact spots. The shaded area is the single continuous contact with the same resistance; the outer circle has the Holm radius of the cluster. Values for contact spots of varying sizes in these positions are given in table 2.

When the self-resistance is large the interaction is close to the approximate value given by equation (5), $(\pi n^2)^{-1} \sum \sum s_{ij}^{-1}$, which in this case equals 0.0941, but as the self-resistance becomes less important the interaction decreases and finally, when the contacts touch, it has almost decreased by the factor $3\pi^2/32$ (which would give 0.0870) corresponding to the change from equation (6) to equation (1).

Table 2. Effect of varying contact size

Contact radius a	Resistance R (from simultaneous eqns)	Self-resistance $1/2na$	Interaction $R - (1/2na)$ (by difference)	Holm radius α (see eqn (8))	Radius of single spot of same resistance
0.02	0.4226	0.3289	0.0937	5.34	1.18
0.04	0.2577	0.1645	0.0932	5.36	1.94
0.1	0.1581	0.0658	0.0923	5.42	3.16
0.2	0.1238	0.0329	0.0909	5.50	4.04
0.5	0.1012	0.0132	0.0880	5.68	4.94

The array spacing is taken as unity.

The variation is small compared to the accuracy of contact resistance values and so may simply be ignored. However, it is more desirable to estimate the interaction correctly when it is the dominant term than when it is small. Thus it is recommended that the interaction term of equation (5) should be reduced by $3\pi^2/32$ and the new value be used whatever the size of the self-resistance. Thus,

$$R = \rho \left(\frac{1}{2\sum a_i} + \frac{3\pi}{32n^2} \sum \sum \frac{1}{s_{ij}} \right). \quad (7)$$

This cannot be derived rigorously by the present approach: it has been found only by combining mathematics and physics. Equation (7), of course, applied to Holm's example of a well-filled circular disk, leads to equation (1) exactly.

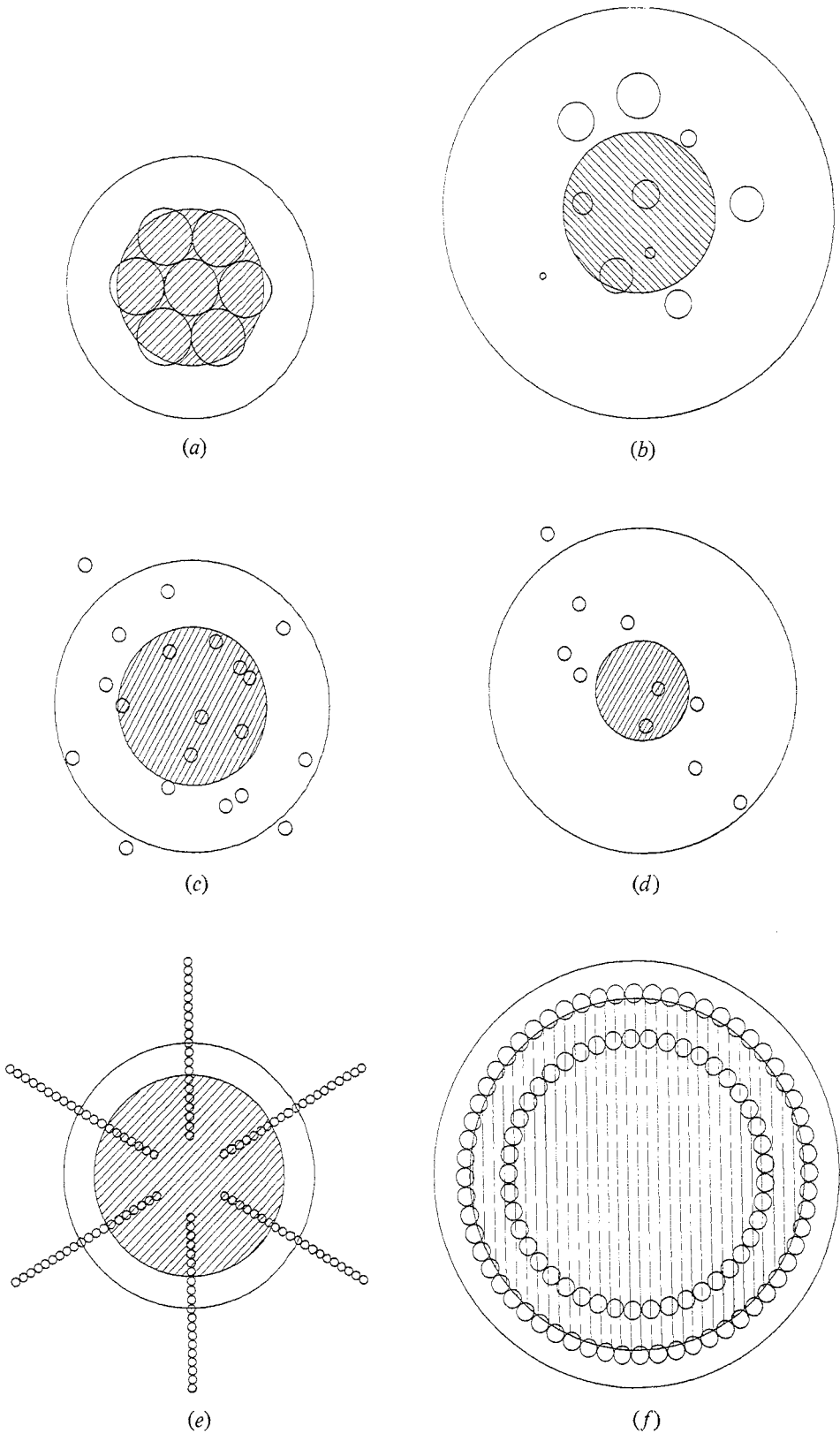


Figure 4. Typical clusters, showing the equivalent single contact (shaded) and the Holm radius.

3.2. The Holm radius of a general cluster

Holm's equation provides a method of obtaining a practical estimate of the interaction term, provided we can establish the value of α for non-uniform non-circular clusters. This may be done by taking a number of clusters whose resistances have been found and examining the value of α required in equation (1) to give these resistances. Thus, we define

$$\alpha = (2R/\rho - 1/\sum a_i)^{-1}. \tag{8}$$

Since the introduction of the term $\rho/2\alpha$ is due to Holm, we shall refer to α as the 'Holm radius' of the cluster. Figure 4 shows examples of clusters and their Holm radii.

When the number of contacts is small they do not form a well-defined cluster and the Holm radius is large. As the number of spots increases, a definite Holm radius emerges, and varies little as the area of the cluster is filled up. This is demonstrated in figure 5, in which spots of random sizes were steadily added to the cluster, at random positions governed

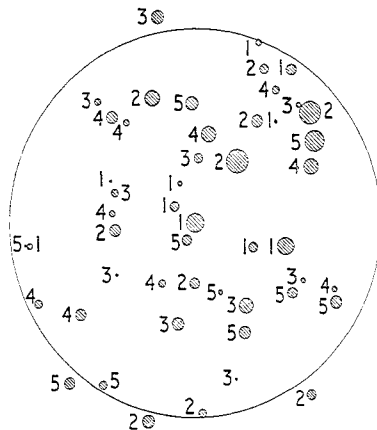


Figure 5. Cluster formed by adding contact spots according to a fixed probability distribution. The first ten are marked 1, the second ten 2, and so on. The large circle indicates the Holm radius for the 50 contacts shown. The table below shows that the Holm radius is almost constant.

Number of contact spots	5	10	15	20	30	40
Holm radius	10.73	8.75	9.17	9.78	9.73	9.71
Number of contact spots	50	60	70	80	90	100
Holm radius	9.76	9.91	9.60	9.57	9.64	9.63

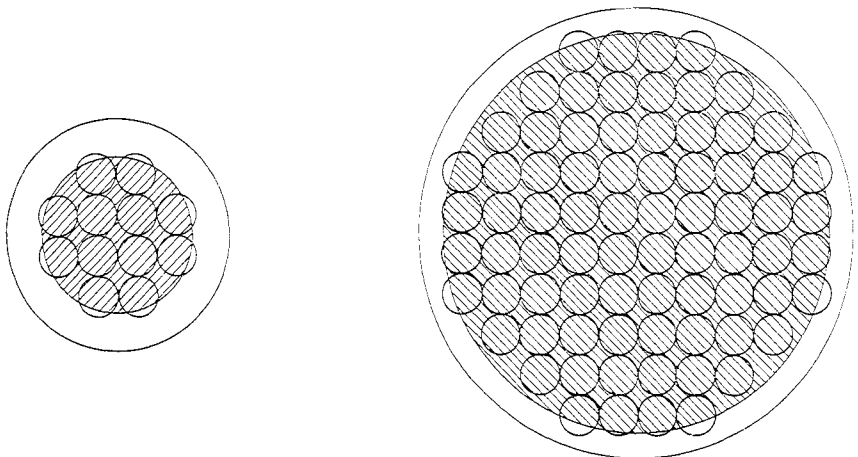


Figure 6. Densely packed clusters, showing the equivalent single contact (shaded) and the Holm radius.

by a fixed statistical distribution. The Holm radii are given after the addition of every 10 spots; they quickly reach a steady value. Figure 6 shows examples of densely packed clusters of different sizes. From these it is clear that the Holm radius is not the envelope of the contacts. A general rule of thumb for an irregular cluster is to draw an envelope lying outside each peripheral spot by a distance equal to the centre-to-centre separation from its nearest neighbour: the Holm radius is approximately that of the circle containing the same area.†

4. Discussion

4.1. Accuracy

We have shown that the resistance of a well-filled cluster of contacts is approximately

$$R = \rho \left(\frac{1}{2n\bar{a}} + \frac{1}{2a} \right)$$

where a , the Holm radius of the cluster, is defined by

$$a^{-1} \equiv \frac{3\pi}{16n^2} \sum \sum \frac{1}{s_{ij}}$$

Holm originally suggested that an equation of this form was accurate to 10% provided $a < a/3n^{1/2}$, which is the condition for the mean separation of the contacts to be greater than three times their radius. This estimate is now shown to be unduly pessimistic. The present treatment contains three sources of error. First, proximity (higher-order) effects are ignored in setting up equation (3). These have been shown not to affect the resistance appreciably, so there is no restriction of the type Holm envisaged. Secondly, there is a systematic variation in the interaction term which depends on whether the self-resistance is major or minor. This affects both equation (4) and equation (5). By using $3\pi/16$ in the definition of a instead of the original $2/\pi$, the resulting error will be negligible, except when the term it affects is small. Thirdly, the removal of the contact radius from the double summation in equation (5) introduces a statistical error when the contact spots are of different sizes. This is small, and becomes even smaller as the number of spots increases. In normal cases the total error due to these approximations will be less than 1%.

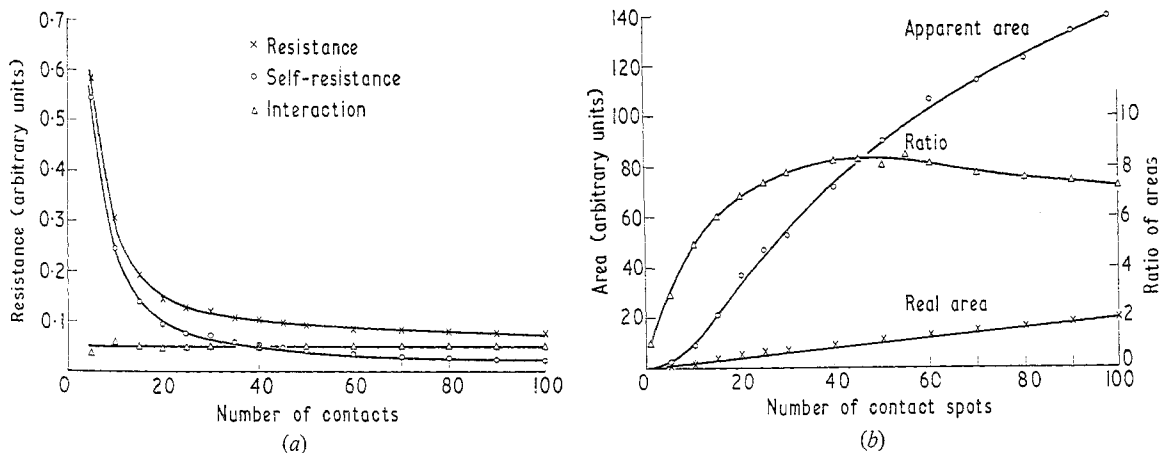


Figure 7. (a) Variation of self-resistance, interaction, and total resistance with number of contact spots for the cluster of figure 5. (b) Comparison of real area of contact with value deduced from the resistance.

† This rule fails when the contacts are distributed along a line or lines as in figures 4(e) and 4(f) instead of over an area. The interaction term is then not independent of the number of contacts and it appears to increase logarithmically as expected theoretically. For example, rings of radius 1.0, containing 30, 60, and 120 contacts, give interaction terms of 0.425, 0.496 and 0.567.

4.2. Constriction resistance and the real area of contact

It is common practice in the study of friction and electric contacts to estimate the area of contact from the measured constriction resistance using the hypothesis of a single circular contact. The results of doing this with the clusters considered here have been shown in figures 3–6, and it is clear that there is no agreement. The results in table 2 are typical: there the resistance decreased by a factor of 4, suggesting—according to the single contact hypothesis—that the area had increased 16 times, whereas in fact it had increased by a factor of 625. The cluster of figure 5, in which the number of contacts was steadily increased, provides another example; figure 7(a) shows the variation of the resistance, and of its two components. Figure 7(b) compares the real area of contact with that deduced from the resistance: the ratio of the two rose rapidly to 8; there were then 50 contacts, and the self-resistance and the interaction had become equal. It then slowly decreased as the addition of further contacts increased the real area without greatly reducing the resistance. With a larger number of smaller spots, the ratio will be greater. Thus it is impossible to deduce anything about the real area of contact directly from measurements of the constriction resistance, except an upper limit; and the actual value can be a small fraction of this.

4.3. Load and the real area of contact

Whenever we have a reasonably large number of not-too-small contacts, the self-resistance term becomes small and the constriction resistance is close to that observed if the entire area of the cluster were in electrical contact. One aspect of this has been discussed before (Cocks 1954 Ph.D. Thesis, Archard 1959): if an insulating film covers the mechanical area of contact, so that electrical contact occurs by a large number of small breaks in the film, the resistance may be almost as low as with no film (compare figure 4(d)). The other aspect has not. The mechanical area of contact is often very much less than that deducible from the bulk properties. The remarkable persistence of surface asperities during bulk plastic flow is well-known (Moore 1948, Greenwood and Rowe 1965): the consequence, that the real area of contact is a small fraction of that predicted from the bulk hardness (see figure 1), is usually ignored. For the present discussion the origin of this extra hardness, whether due to an abnormal state of work-hardening, or to frictional restraint, or to a strengthening effect due to a surface film blocking the movement of dislocations, is irrelevant: the important thing is that the use of the bulk hardness is wrong, and quite badly so. The present work shows why this has remained undetected: the areas of contact found from the constriction resistance, from the bulk hardness, or from visual observation of the contact impression all agree (Bowden and Tabor 1954). From the point of view of electrical contact study this is highly satisfactory: the resistance can be calculated in the usual way from the load, hardness and resistivity. The implications for the study of friction are less satisfactory, for presumably the friction force is proportional to the *real* area of contact. The dilemma has arisen before: Parker and Hatch (1950), sliding lead spheres against a glass surface, identified the *visual* area of contact (which was proportional to the load) as an *apparent* area, and were forced to conclude that for some obscure reason the real and apparent areas were proportional. It now appears that this was not an isolated case. On the contrary, it is doubtful whether there is *any* experimental evidence that the real area of contact is proportional to the load, but rather good experimental evidence that the apparent area is. If the former is still true (and it remains a most persuasive explanation of Amontons' laws), then Parker and Hatch's hypothesis must be of very general validity.

Acknowledgments

I should like to thank Dr. J. H. Tripp and Dr. J. B. P. Williamson for their advice and assistance.

Appendix

The interaction of two circular contacts

We shall consider the electrostatic problem of two coplanar circular conducting disks, radii a and b , carrying charges Q_a and Q_b . The potentials are conveniently found by superposition of the solutions for one charged disk and one uncharged disk: these solutions are found approximately as follows.

In the absence of disk b , a charge Q on a will give a potential distribution, in the plane of the disk, of

$$\frac{Q}{a} \sin^{-1} \left(\frac{a}{s} \right) \sim \frac{Q}{s} + \frac{Qa^2}{6s^3} + O \left(\frac{a^4}{s^5} \right)$$

where as usual $O(\phi)$ denotes a function f such that $|f| \leq K\phi$ when ϕ approaches zero. This may be regarded as due to a point charge Q and two mutually perpendicular linear quadrupoles of magnitude $\frac{1}{6}Qa^2$. (A quadrupole Q' gives a potential $Q'(3 \cos^2 \theta - 1)/s^3$ at an angle θ to its axis: two perpendicular ones therefore produce

$$Q'(3 \cos^2 \theta - 1 + 3 \sin^2 \theta - 1)/s^3 = Q'/s^3$$

in all directions.)

Now a point charge Q distant s from the centre of an uncharged disk gives rise to a potential $(Q/b) \sin^{-1}(b/s)$ on the disk, and also induces a charge distribution on the disk. The potential of the second disk due to the distributed charge Q is therefore approximately

$$\frac{Q}{s} + \frac{1}{6} \frac{Qb^2}{s^3} + O \left(\frac{b^4}{s^5} \right) + \frac{1}{6} \frac{Qa^2}{s^3} + O \left(\frac{a^4}{s^5} \right).$$

The potential of the first disk will be the sum of $Q\pi/2a$ for the isolated disk and the potential due to the induced charges on the second disk. We shall calculate this approximately by assuming the disks so far apart that the field E at the second disk is approximately uniform and equal to Q/s^2 . The charge distribution induced by E is (Smythe 1950, p. 161)

$$\frac{E\rho \cos \phi}{\pi^2(b^2 - \rho^2)^{1/2}}$$

where ρ, ϕ are polar coordinates centred on the second disk.

This distribution has a dipole moment of

$$m = \int_{\phi=0}^{2\pi} \int_{\rho=0}^b \sigma \rho \cos \phi \rho d\rho d\phi = \frac{4}{3\pi} Eb^3 = \frac{4Qb^3}{3\pi s^2}$$

acting along the line of centres (and also higher moments which we neglect). The potential at the centre of the first disk due to this dipole is $-m/s^2$, i.e. $-4Qb^3/3\pi s^4$, and so the total potential at the first disk is

$$\frac{Q}{a} \frac{\pi}{2} - \frac{4}{3\pi} \frac{Qb^3}{s^4}.$$

Finally we superpose two solutions to get the case of both disks charged:

$$\phi_a = Q_a \left(\frac{\pi}{2a} - \frac{4}{3\pi} \frac{b^3}{s^4} \right) + Q_b \left(\frac{1}{s} + \frac{a^2 + b^2}{6s^3} \right) + O(s^{-5}).$$

From this and the corresponding expression for ϕ_b we may solve to find values of Q_a and Q_b which make $\phi_a = \phi_b$, and hence we obtain the capacity of a pair of linked coplanar disks. The result is complicated unless $a = b$; when it is,

$$C^{-1} = \frac{1}{2} \left(\frac{\pi}{2a} + \frac{1}{s} + \frac{a^2}{3s^3} - \frac{4a^3}{3\pi s^4} + \dots \right)$$

and the constriction resistance of two equal circular contact spots must be $(\rho/\pi)C^{-1}$.

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